SCIENCE

MATRICES

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Household energy use

Record weekly energy usage for your household (H1):

Type	quantity	units
• electricity	140	kWh
• gas	50	units
• petrol	40	L

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Household energy use

Record weekly energy usage for multiple households (H1-H4) over two weeks:

Household	Week 1			Week 2		
	electricity	gas	petrol	electricity	gas	petrol
H1	140	50	40	160	50	20
H2	100	20	60	120	40	40
H3	180	0	50	220	0	50
H4	80	80	40	80	120	20

Tables rapidly become complicated/convoluted!
How should a computer handle such tables?

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Create a matrix

Household	Week 1		
	electricity	gas	petrol
H1	140	50	40
H2	100	20	60
H3	180	0	50
H4	80	80	40

Let E be a matrix that shows energy use of H1-4 for 1 week

$$E_1 = \begin{pmatrix} 140 & 50 & 40 \\ 100 & 20 & 60 \\ 180 & 0 & 50 \\ 80 & 80 & 40 \end{pmatrix}$$

This is a 4 x 3 matrix (4 rows, 3 columns)
(4 households, 3 energy types)

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Matrix operations

Transform data to matrices

Household	Week 1			Week 2		
	electricity	gas	petrol	electricity	gas	petrol
H1	140	50	40	160	50	20
H2	100	20	60	120	40	40
H3	180	0	50	220	0	50
H4	80	80	40	80	120	20

$$E_1 = \begin{pmatrix} 140 & 50 & 40 \\ 100 & 20 & 60 \\ 180 & 0 & 50 \\ 80 & 80 & 40 \end{pmatrix} \quad E_2 = \begin{pmatrix} 160 & 50 & 20 \\ 120 & 40 & 40 \\ 220 & 0 & 50 \\ 80 & 120 & 20 \end{pmatrix}$$

What is total energy use (T)?

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Matrix addition (and subtraction)

What is total energy use (T)?

$$T = E_1 + E_2 = \begin{pmatrix} 140 & 50 & 40 \\ 100 & 20 & 60 \\ 180 & 0 & 50 \\ 80 & 80 & 40 \end{pmatrix} + \begin{pmatrix} 160 & 50 & 20 \\ 120 & 40 & 40 \\ 220 & 0 & 50 \\ 80 & 120 & 20 \end{pmatrix}$$

$$= \begin{pmatrix} 140+160 & 50+50 & 40+20 \\ 100+120 & 20+40 & 60+40 \\ 180+220 & 0+0 & 50+50 \\ 80+80 & 80+120 & 40+20 \end{pmatrix}$$

$$= \begin{pmatrix} 300 & 100 & 60 \\ 220 & 60 & 100 \\ 400 & 0 & 100 \\ 160 & 200 & 60 \end{pmatrix}$$

Matrices must be same size

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Scalar multiplication

What is average energy use (A)?

$$A = \tfrac{1}{2} T = \tfrac{1}{2} \times \begin{pmatrix} 300 & 100 & 60 \\ 220 & 60 & 100 \\ 400 & 0 & 100 \\ 160 & 200 & 60 \end{pmatrix}$$

$$= \begin{pmatrix} \tfrac{1}{2} \times 300 & \tfrac{1}{2} \times 100 & \tfrac{1}{2} \times 60 \\ \tfrac{1}{2} \times 220 & \tfrac{1}{2} \times 60 & \tfrac{1}{2} \times 100 \\ \tfrac{1}{2} \times 400 & \tfrac{1}{2} \times 0 & \tfrac{1}{2} \times 100 \\ \tfrac{1}{2} \times 160 & \tfrac{1}{2} \times 200 & \tfrac{1}{2} \times 60 \end{pmatrix}$$

$$= \begin{pmatrix} 150 & 50 & 30 \\ 110 & 30 & 50 \\ 200 & 0 & 50 \\ 80 & 100 & 30 \end{pmatrix}$$

Scalar applied to whole matrix

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Matrix manipulations

Convert energy usage into greenhouse gas emissions (kg CO_2)

1 kWh electricity produces 1 kg CO_2

1 unit gas produces 0.4 kg CO_2

1 L petrol produces 2.2 kg CO_2

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Convert E table to G emissions

Household	Average usage		
	electricity	gas	petrol
H1	150	50	30
H2	110	30	50
H3	200	0	50
H4	80	100	30

Calculate greenhouse gas emissions (G) for H1:

G1 = (150 kWh x 1 kg/kWh) + (50 units x 0.4 kg/unit) + (30 L x 2.2 kg/L)

= 150 kg + 20 kg + 66 kg

= 236 kg

Laborious to do them all

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Use matrix multiplication

Conversions can be given in a 3 x 1 matrix (F)

1 kWh electricity produces 1 kg CO_2

1 unit gas produces 0.4 kg CO_2

1 L petrol produces 2.2 kg CO_2

$$\begin{pmatrix} 1 \\ 0.4 \\ 2.2 \end{pmatrix} = F$$

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Matrix multiplication

Calculate average weekly greenhouse gas emissions

$$G = A \times F = \begin{pmatrix} 150 & 50 & 30 \\ 110 & 30 & 50 \\ 200 & 0 & 50 \\ 80 & 100 & 30 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0.4 \\ 2.2 \end{pmatrix}$$

add products of columns in A by rows in F

$$= \begin{pmatrix} (150 \times 1) + (50 \times 0.4) + (30 \times 2.2) \\ (110 \times 1) + (30 \times 0.4) + (50 \times 2.2) \\ (200 \times 1) + (0 \times 0.4) + (50 \times 2.2) \\ (80 \times 1) + (100 \times 0.4) + (30 \times 2.2) \end{pmatrix} = \begin{pmatrix} 236 \\ 232 \\ 310 \\ 186 \end{pmatrix}$$

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Matrix multiplication

matrix multiplication better visualized as follows:

$$\begin{pmatrix} 150 & 50 & 30 \\ 110 & 30 & 50 \\ 200 & 0 & 50 \\ 80 & 100 & 30 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0.4 \\ 2.2 \end{pmatrix} = \begin{pmatrix} 236 \\ 232 \\ 310 \\ 186 \end{pmatrix}$$

multiply 4 x 3 matrix by 3 x 1 matrix gives 4 x 1 matrix

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MATRICES

Very clever mathematics

Can handle large data sets

Can handle complex manipulations

Ideal for computerization

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Matrix operations

Must conform to specific set of rules

- matrix multiplication — $(m \times n) \times (n \times q) = (m \times q)$
- order of operations — $(AB \neq BA)$
- identity matrices — $(A \times I = A)$
- inverse matrices — (If $AX = B$, then $X = A^{-1}B$)

Let us examine these operations in another context;

that of population biology,

involving simultaneous equations

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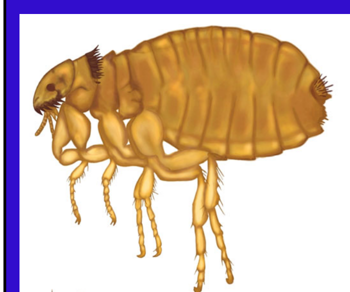
Exemplar: age/stage population structure

Fleas (itchy-scratchy syndrome)




the dangers of not wearing shoes!

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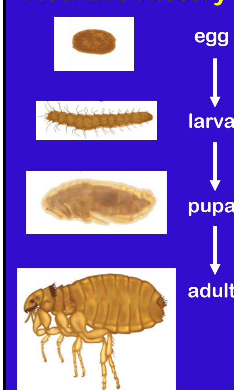
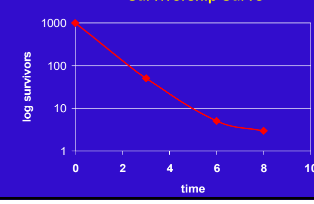
Flea Life History



Life Table

Age (weeks)	Dev. Stage	Number alive	Proportion alive
0-2	eggs	10,000	1.00
3-5	larvae	500	0.05
6-7	pupae	50	0.005
8-20	adults	30	0.003

Survivorship Curve



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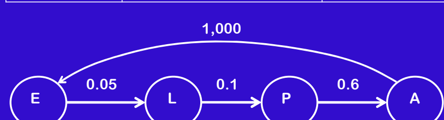
Stage-structured diagrams

- used to depict life-cycles
- arrows show proportion in transition

Time period	Flea developmental stage	Number alive
1	Eggs	10,000
2	Larvae	500
3	Pupae	50
4	Adults (50% female) (females lay 2,000 eggs)	30



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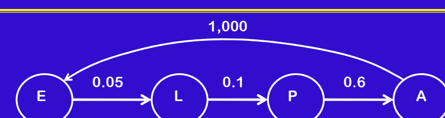
Simultaneous equations



$$E_{i+1} = 0\,E_i + 0\,L_i + 0\,P_i + 1{,}000\,A_i = 1{,}000\,A_i$$

$$L_{i+1} = 0.05\,E_i + 0\,L_i + 0\,P_i + 0\,A_i = 0.05\,E_i$$

$$P_{i+1} = 0\,E_i + 0.1\,L_i + 0\,P_i + 0\,A_i = 0.1\,L_i$$

$$A_{i+1} = 0\,E_i + 0\,L_i + 0.6\,P_i + 0\,A_i = 0.6\,P_i$$

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Matrix operation

$$\begin{aligned} E_{i+1} &= \Big(0\,E_i + 0\,L_i + 0\,P_i + 1{,}000\,A_i \\ L_{i+1} &= 0.05\,E_i + 0\,L_i + 0\,P_i + 0\,A_i \\ P_{i+1} &= 0\,E_i + 0.1\,L_i + 0\,P_i + 0\,A_i \\ A_{i+1} &= 0\,E_i + 0\,L_i + 0.6\,P_i + 0\,A_i \Big) \end{aligned}$$

transition matrix

$$\begin{pmatrix} 0 & 0 & 0 & 1{,}000 \\ 0.05 & 0 & 0 & 0 \\ 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0.6 & 0 \end{pmatrix}$$

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Matrix multiplication

The current population (N_1) comprises
100 eggs (E), 50 larvae (L),
10 pupae (P), and 2 adults (A)

Expressed as matrix:

$$\begin{pmatrix} 100 \\ 50 \\ 10 \\ 2 \end{pmatrix}$$

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Matrix multiplication

calculate N_2 (move forward in time)

$$\begin{pmatrix} 0 & 0 & 0 & 1{,}000 \\ 0.05 & 0 & 0 & 0 \\ 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0.6 & 0 \end{pmatrix} \times \begin{pmatrix} 100 \\ 50 \\ 10 \\ 2 \end{pmatrix} = \begin{pmatrix} 2{,}000 \\ 5 \\ 5 \\ 6 \end{pmatrix}$$

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Retrospective calculations

Another ectoparasite of dogs has only two life-cycle stages (juveniles and adults) which cycle monthly. The transition matrix for this parasite is:

$$\begin{pmatrix} 0.4 & 2 \\ 0.5 & 0.2 \end{pmatrix}$$

There were 44 juveniles and 9 adults on your dog in July.
How many were there in June?
(i.e. the previous month)

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Inverse matrix

To solve matrix equation,
need to calculate inverse matrix

When $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $A^{-1} = \dfrac{1}{(ad-bc)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

When $A = \begin{pmatrix} 0.4 & 2 \\ 0.5 & 0.2 \end{pmatrix}$, $A^{-1} = \dfrac{1}{(-0.92)} \begin{pmatrix} 0.2 & -2 \\ -0.5 & 0.4 \end{pmatrix}$

WHY?

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Solving matrix equations

The solution to $AX = B$ is $X = A^{-1}B$ (if A^{-1} exists)

$A = \begin{pmatrix} 0.4 & 2 \\ 0.5 & 0.2 \end{pmatrix}$, $X = P_{June} = \begin{pmatrix} x \\ y \end{pmatrix}$, $B = P_{July} = \begin{pmatrix} 44 \\ 9 \end{pmatrix}$

$$\text{so } P_{June} = \frac{1}{(-0.92)} \begin{pmatrix} 0.2 & -2 \\ -0.5 & 0.4 \end{pmatrix} \times \begin{pmatrix} 44 \\ 9 \end{pmatrix}$$

$$= \frac{1}{(-0.92)} \begin{pmatrix} -9.2 \\ -18.4 \end{pmatrix} = \begin{pmatrix} 10 \\ 20 \end{pmatrix}$$

so there were 10 juveniles ands 20 adults in June!

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