

SCIE1000, Solutions to Tutorial Week 10.

- (1) 1. We substitute each given value of T into the formula

$$L = 10^{(T-121)/10}.$$

When $T = 111$, $L = 10^{-1} = 0.1$. When $T = 115$, $L = 0.251$. When $T = 118$, $L = 0.501$.

2. It might help first to create a table showing L values over time.

time t (mins)	0	1	2	3	4	5	6
L	0.1	0.251	0.501	0.501	0.501	0.251	0.1

Now the answer is a simple application of the trapezoid rule to the data in that table. The AUC for each trapezoid, in turn, is:

$$(1 - 0) \times (0.1 + 0.251)/2 = 0.176$$

$$(2 - 1) \times (0.251 + 0.501)/2 = 0.376$$

$$1 \times 0.501 = 0.501$$

$$1 \times 0.501 = 0.501$$

$$1 \times (0.501 + 0.251)/2 = 0.376$$

$$1 \times (0.251 + 0.1)/2 = 0.176$$

Adding these gives an F -value of 2.106.

3. The process is not safe: it needs an F -value of at least $F_0 = 2.4$.
4. Note that $L = 1.0$ at $T = 121$. The area of the third trapezoid is now $1 \times (0.501 + 1)/2 = 0.75$. The fourth trapezoid has the same area. The new F -value is $F = 2.604$ and the process is likely to be safe.
5. The output is:

Val: 2 116.5

Val: 3 118.0

Val: 4 116.5

End: 5 116.5

6. Here is a solution:

```
# Calculate L values
from pylab import *

def calc(T):
    v = 10**((T-121)/10)
    return v

t = arange(0,7)
Temp = array([111, 115, 118, 118, 115, 111])
i = 0
while i<6:
    m = calc( Temp[i] )
    print("Temp: ",Temp[i], "L value: ", m)
    i = i+1
```

- (2) This is a discussion question.
- (3) The first thing to notice about these data is that as temperature increases, time decreases. You could sketch a **really** rough graph to see what the relationship looks like, but a big give-away about what model to use for this data is to notice that, while the temperature is going up by a constant rate, the time taken to cook is approximately halving (this is especially obvious at temperatures from 126–132). Data which have a constant halving (or doubling) time is best modelled by an exponential function. From the graph, the approximate halving “time” (really, halving temperature) is 2 degrees.

Because the time is reducing (decaying) as temperature is increased, we will use a model of the form $t(f) = Ae^{-kf}$.

Using a halving “time” of 2 degrees, we have

$$\begin{aligned} 1/2 &= e^{-k \times 2} \\ \ln(1/2) &= -2k \\ k &\approx 0.346 \end{aligned}$$

In the table, the temperature f starts at 120, and goes up. As a trick, we can let $g = (f - 120)$ in the equation, so the value of g starts at 0 and goes up. So when $g = 0$, the time is 1260 minutes.

Thus we have

$$\begin{aligned} t &= Ae^{-0.346g} \\ \text{so } 1260 &= Ae^{-0.346 \times 0} \\ \text{so } A &= 1260. \end{aligned}$$

Hence the equation is $t = 1260e^{-0.346g}$ or $t = 1260e^{-0.346(f-120)}$.

- (4)
1. The area under the graph represents total exposure to temperatures over 60 °C, and its units are min °C.
 2. We can calculate the exact area under the graph by using the trapezoidal rule. We have four distinct phases of heating, 0-10 mins, 10-40 mins, 40-50 mins and 50-60 mins. The times 0, 10, 40, 50 and 60 minutes will be referred to as t_0 to t_4 , and the temperature at time t_i will be f_i . By the Trapezoid rule, the area under the curve will be

$$\begin{aligned} AUC &= (t_1 - t_0) \frac{f_0 + f_1}{2} + (t_2 - t_1) \frac{f_1 + f_2}{2} + (t_3 - t_2) \frac{f_2 + f_3}{2} + (t_4 - t_3) \frac{f_3 + f_4}{2} \\ &= (10 - 0) \frac{0 + 10}{2} + (40 - 10) \frac{10 + 10}{2} + (50 - 40) \frac{10 + 8}{2} + (60 - 50) \frac{8 + 0}{2} \\ &= 10 \frac{10}{2} + 30 \frac{20}{2} + 10 \frac{18}{2} + 10 \frac{8}{2} \\ &= 50 + 300 + 90 + 40 \\ &= 480 \text{ min } ^\circ\text{C}. \end{aligned}$$

3. From our work in the last part of the question, the area under the curve for the last 50 mins of the process is $300 + 90 + 40 = 430$ min °C. We need the area under the curve for the first 10 minutes to be exactly 70 min °C, and the only thing we can change is the initial heating rate. Graphically, what we can change is the time at which the oven reaches 70 °C. Let this time be t_{70} . We can model the area under the graph with two trapezoids, one from 0 minutes to time t_{70} and one from time t_{70} to 10 minutes (the temperatures above 60 °C at these times are 0, 10 and 10 respectively in °C). Using the trapezoid rule again gives

$$\begin{aligned} AUC &= (t_{70} - 0) \frac{0 + 10}{2} + (10 - t_{70}) \frac{10 + 10}{2} \\ &= 5t_{70} + 10(10 - t_{70}) \\ &= 5t_{70} + 100 - 10t_{70} \\ &= 100 - 5t_{70}. \end{aligned}$$

Thus, if we want the area under this part of the curve to be 70 min °C, we can find t_{70} by solving

$$\begin{aligned} 70 &= 100 - 5t_{70} \\ 70 + 5t_{70} &= 100 \\ 5t_{70} &= 100 - 70 \\ t_{70} &= \frac{30}{5} \\ t_{70} &= 6. \end{aligned}$$

For the total area under the curve to be 500 min °C, we need the initial heating to be completed in 6 minutes, which corresponds to an initial rate of heating of $\frac{10}{6} = 1.667$ °C min⁻¹.

- (5) 1. From the hint, it is clear that we should use a sine curve to model this function. If a sine curve starts with no phase shift, its initial value is also its centre value. Furthermore, for the same sine curve, after a quarter of a period, the curve reaches its maximum amplitude. From this and the graph, we can deduce that the centre value is 0 and the amplitude is 10. We also know that a quarter of the period took 10 minutes, so the whole period of this sine curve is 40 minutes. Putting all this together gives the sine model

$$T(t) = 10 \sin\left(\frac{2\pi t}{40}\right).$$

2. There are 11 cells completely under the graph defined by the solid curve, with 3 cells almost completely under the graph, 4 cells about half under the graph and 1 cell less than half under the graph. If we assume the less than half filled cell completes the 3 almost complete cells, then we have roughly $11+3+4/2=16$ cells under the graph. Each cell has an area of $4 \text{ min } ^\circ\text{C}$, so the area under the solid graph is roughly $4 \times 16 = 64 \text{ min } ^\circ\text{C}$.

The area under the dashed graph can be calculated using the formula for the area of a triangle, as $0.5 \times 10 \times 10 = 50 \text{ min } ^\circ\text{C}$.

Thus, the difference between the areas is about $14 \text{ min } ^\circ\text{C}$.

3.

$$AUC = \frac{AP}{2\pi} = \frac{10 \times 40}{2\pi} = \frac{200}{\pi} = 63.662 \text{ min } ^\circ\text{C}$$

Thus the true difference between the areas is $63.662 - 50 = 13.662 \text{ min } ^\circ\text{C}$.

$$\% \text{ error} = \frac{14 - 13.662}{13.662} \times 100 = 2.47$$

So the percent error in our rough estimate is only 2.47%.

- (6) One possible solutions is:

```
from pylab import *

def trapArea(y1,y2,wid):
    return (y1+y2)/2*wid

t=arange(120,144,2)
T=array([1260,570,270,120,60,30,15,6,3,2,1,1])
numTemp=12
plot(t,T,'x')
i=0
area=0
while i<numTemp-1:
    a=trapArea(T[i],T[i+1],2)
    area=area+a
    i=i+1
print(area)
show()
```

- (7) 1. The answer is:

$$C = 2 \times (1000 + 800)/2 + 6 \times 800 + 2 \times 1000 + 2 \times (1000 + 1200)/2 = 11000$$

So $C = 11000 \text{ Watt hours}$.

2. Here is the output.

```
a[i] = 2000.0
a[i] = 4800.0
a[i] = 1600.0
a[i] = 2000.0
AUC = 10400.0 units.
```

3. The program does not handle the “triangular” parts of the graph correctly, at $t = 0$ and $t = 10$. The program would need to use the trapezoid rule at those times. There is also an error at $t = 8$ when the usage changes according to a step function; there would need to be an additional entry in the array at this time.
4. Here is a possible solution. Areas are calculated as trapezoids rather than rectangles. Also, an extra entry was added to the array for both time and power usage, to represent the step change in power usage at time $t = 8$. This has created a trapezoid of “zero width”, which does not affect the calculated area, but does mean that the rest of the calculations are now correct.

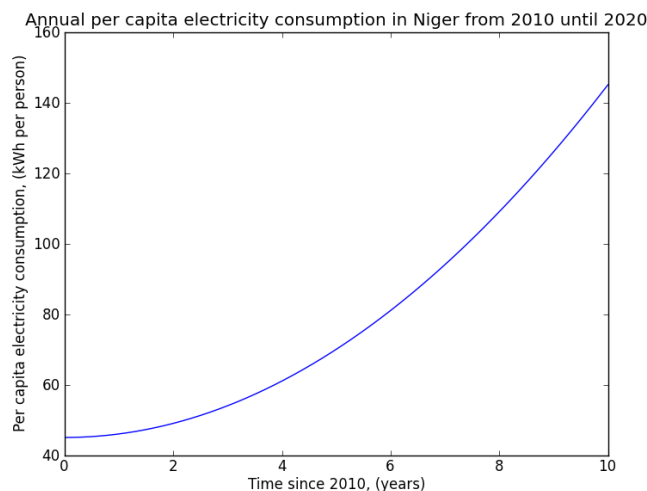
```
# Program to calculate AUC
from pylab import *

t = array([0, 2, 8, 8, 10, 12])
p = array([1000, 800, 800, 1000, 1000, 1400])
tA = 0
a = zeros(6)
i = 0
while i<5:
    a[i] = (p[i+1] + p[i])/2 * (t[i+1] - t[i])
    tA = tA + a[i]
    print("a[i] = ",a[i])
    i = i + 1
print("AUC = ", tA, "units. ")
```

Here is the output; it matches the AUC value calculated on the tutorial sheet.

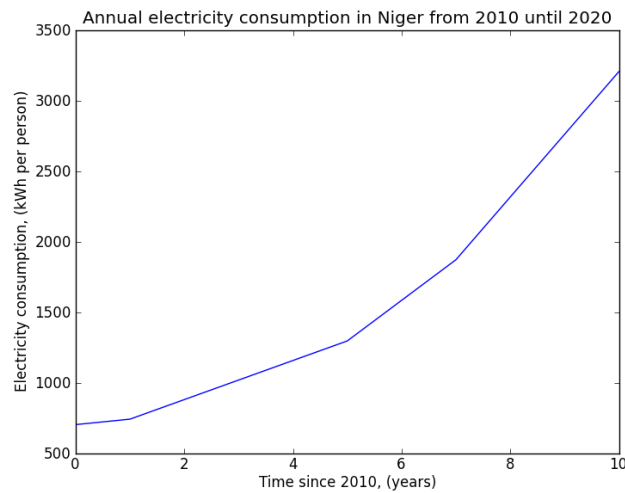
```
a[i] = 1800.0
a[i] = 4800.0
a[i] = 0.0
a[i] = 2000.0
a[i] = 2400.0
AUC = 11000.0 units.
```

- (8) The author is claiming to have discovered the trapezoid rule! In other words, she is publishing a paper that claims to discover a method that has in fact been known for centuries! This is worrying on many levels. How could she not realise that this approach is well known? How could the journal referees not recognise this? When she talks about other methods that are less accurate, presumably she is referring to something similar to the trapezoid rule... if so, how can they be less accurate? It's all rubbish!
- (9) 1. Here is the graph:



2. The following table summarises the calculations required in this question. Calculating the per capita usage column is just a matter of subbing the corresponding value of t into the formula for $e(t)$. The last column is just the per capita usage divided by the population at that time.

Year	t	Population of Niger	Per capita usage (kWh per person)	Total annual electricity consumption for Niger (million kWh)
2010	0	15.6	45	702
2011	1	16.1	46	740.6
2015	5	18.5	70	1295
2017	7	19.9	94	1870.6
2020	10	22.1	145	3204.5



3.

4. The gradient is “the rise over the run”, so the units of the gradient will be the units of the y -axis over the units of the x -axis. For this graph, the units of the gradient are kWh/year.

The units of the area under the curve is the product of the units of the axes of the graph, which for this graph will be kWh years.

5. The exact area under the graph can be found by using the trapezoid rule.

$$\begin{aligned}
 AUC &= (t_1 - t_0) \frac{f_0 + f_1}{2} + (t_2 - t_1) \frac{f_1 + f_2}{2} + (t_3 - t_2) \frac{f_2 + f_3}{2} + (t_4 - t_3) \frac{f_3 + f_4}{2} \\
 &= (1 - 0) \frac{702 + 740.6}{2} + (5 - 1) \frac{740.6 + 1295}{2} + (7 - 5) \frac{1870.6 + 1295}{2} + (10 - 7) \frac{1870.6 + 3204.5}{2} \\
 &= \frac{1442.6}{2} + 2(2035.6) + (3165.6) + 3 \frac{5075.1}{2} \\
 &= 721.3 + 4071.2 + 3165.6 + 7612.65 \\
 &= 15570.75
 \end{aligned}$$

So the exact AUC of the graph is 15570.75 kWh years.

- (10) 1. The strength of the Fundamental Theorem of Calculus is that it is 100% precise, in that if you know the function and the bounds on the integral, the final result is the exact, ‘true’ value. The primary weakness is that the function which needs to be integrated will often not be known.

The strength of the ‘area by rectangles’ method lies in its versatility to be used when we do not have a specified, known function. It can be (and is) used for a wide variety of applications, especially when we

are sampling data at various time intervals and the underlying function we wish to integrate is either very difficult or impossible to determine exactly. The weaknesses of this method is that it may sacrifice accuracy for applicability - it is almost always an approximation to the true value of the area under the curve (or data), and unless we are able to obtain the underlying function we may not know just how accurate the 'area by rectangles' method is.

2. To work out the area underneath the data we multiply the concentration values by the corresponding difference in adjacent time measurements, before summing all of the products. Therefore:

$$\begin{aligned}\text{Area} &= 0 \times (0.1 - 0) + 2 \times (0.5 - 0.1) + 4 \times (1.2 - 0.5) + 3 \times (1.4 - 1.2) \\ &\quad + 0 \times (1.7 - 1.4) + 2 \times (2.0 - 1.7) \\ &= 0 + 2 \times 0.4 + 4 \times 0.7 + 3 \times 0.2 + 0 + 2 \times 0.3 \\ &= 0.8 + 2.8 + 0.6 + 0.6 = 4.8 \text{ mmol. hr/L}\end{aligned}$$

The physical meaning of the value 4.8 mmol.hr/L is effectively the 'exposure' of the person to the drug.