SCIE1000, Solutions to Tutorial Week 11.

(1)

1. Answers will vary.
2. All of these models could be parts of a logistic model, with different values of growth rate and carrying capacity. However, if we are not trying to model the parts of the graph we can't see, then there are simpler functions we can use to model these graphs.

   The constant fertility model is made by assuming exponential growth will occur, with no consideration of a global carrying capacity for humanity. The high fertility rate model appears to be increasing at a constant rate, so a linear model could be used to approximate this prediction. The medium fertility rate model has a classic elongated 's' shape which a logistic model describes, so the logistic model would be a good model to approximate this prediction. The low fertility growth rate has a single peak, so it may be possible to model this graph with a parabola.

(2) Answers to this question will vary, depending on what day you view the clock, and what year it is. Here are example answers, for the year 2014, that show how the calculations can be done.

1. On February 6th 2014 there had been about 14 million births and 5.8 million deaths so far in 2014. This is the 37th day of the year. So by the end of the year the total number of births is estimated as $365/37 \times 14$ million, which is 138 million. The total number of deaths is estimated as $365/37 \times 5.8$ million, which is 57 million. The net increase in population is thus estimated to be $138 - 57 = 81$ million.
2. Over a period of 1 minute, the population of China rose by 17 and the population of India by 29, so India was growing more quickly than China by 12 people per minute. There are 1440 minutes in a day, so the daily difference is $12 \times 1440 = 17280$ people.
3. These answers will vary.
4. Here is the output:

```
2 2.4
3 1.4
```

(3)

1. This is a discussion question.
2. This is a discussion question.
3. This is a discussion question.

(4)

1. This DE means that the rate at which the population changes at any point in time is proportional to how many individuals there are at that point in time. The physical meaning of the constant k is the "net growth rate" of the algae.
2. Taking the derivative of both sides of the equation: $P' = k \times A\,e^{kt} = kP$.

   Thus, this solution satisfies the DE. It means that the growth of the algae over time is exponential in nature, with the constant A representing the initial number of algae.
3. Substituting in the values for $t = 2\,\text{hours}$:

$$\begin{aligned} & P(2) = 200 \\ \Longrightarrow \quad & A\,e^{2k} = 200 \\ \Longrightarrow \quad & 2A\,e^{2k} = 400 \end{aligned}$$

   For $t = 6\,\text{hours}$:

$$\begin{aligned} & P(6) = 400 \\ \Longrightarrow \quad & A\,e^{6k} = 400 \end{aligned}$$

   Equating these two equations:

$$\Longrightarrow \quad 2A\,e^{2k} = A\,e^{6k}$$

Since $A \neq 0$:

$$\begin{aligned}
\implies \qquad 2 &= \frac{e^{6k}}{e^{2k}} \\
&= e^{4k} \\
\implies \qquad \ln(2) &= 4k \\
\implies \qquad k &= \frac{\ln(2)}{4} \\
&\approx 0.173\,\text{hour}^{-1} \\
\implies A\,e^{2\times 0.173} &= 200 \\
\implies \qquad A &= \frac{200}{e^{2\times 0.173}} \\
&= 142\,\text{individuals/mL}
\end{aligned}$$

Thus, an equation for $P(t)$ is:

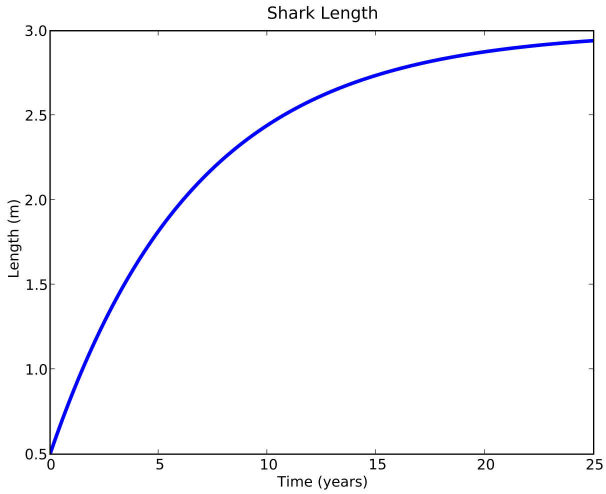
$$P(t) = 142 \times e^{0.173t}$$

4. This will not be realistic over long periods of time, as the algae population will eventually run out of food and space. The current model is exponential; a more realistic model will be logistic (with its characteristic S-shaped curve).
5. This DE means that the rate at which the population changes at any point in time is proportional to how many individuals there are at that point in time (with an intrinsic "net growth rate" k), and also proportional to 'how far' the population is from the carrying capacity T. The term $1 - P/T$ represents competition for resources in the habitat.
6. If the initial population is $2T$ then $1 - P/T = 1 - 2T/T = 1 - 2 = -1$. then the original DE becomes $P' = -kP$, which is the equation for exponential decay. Hence the population will decrease.
7. The initial population is 100. Hence at $t = 0$, $P' = 0.1 \times 100 \times (1 - 100/500) = 8$. Hence at time $t = 0.5$, $P = 100 + 0.5 \times 8 = 104$.

(5)

1. The DE is: $L' = r(M - L)$.
2. In the equation in Part 1, if the length L is larger than M then $M - L$ would be negative, so the rate of change in length is negative, so the shark would reduce in length.
3. Consider the function $L(t) = M - (M - L_0)e^{-rt}$. From the hint,
   $L'(t) = -r \times -(M - L_0)e^{-rt} = r(M - L_0)e^{-rt}$.
   Now consider the DE from Part 1, $L' = r(M - L) = rM - rM + r(M - L_0)e^{-rt} = r(M - L_0)e^{-rt}$.
   Because these last two expressions are identical, the given function $L(t)$ satisfies the DE from Part 1.
4. The solution is $L(t) = M - (M - L_0)e^{-rt}$. If $L_0 = M/2$ then $L(t) = M - (M - M/2)e^{-rt}$. Then when $L = M$ we have
   $M = M - (M - M/2)e^{-rt}$, which simplifies to $0 = M/2e^{-rt}$, or $e^{-rt} = 0$. But this never happens, as the exponential function never equals 0. Hence the shark **never** reaches length M.
5. A. Substituting in values gives: $L(t) = 3 - (3 - 0.5)e^{-0.15t} = 3 - 2.5e^{-0.15t}$.
   To find when $L(t) = 2$, we have: $2 = 3 - 2.5e^{-0.15t}$ so $1 = 2.5e^{-0.15t}$, so $0.4 = e^{-0.15t}$.
   Taking $\ln$ of each side gives: $-0.916 = -0.15t$ and solving gives $t \approx 6.11$ years.

   B. Here are some points on the graph: when $t = 0$ years, $L = 0.5$ m. When $t = 6.11$ years, $L = 2$ m. The maximum length is 3 m, so the graph must gradually approach 3m as t gets large. This is enough info to draw a rough sketch. Here is a computer-generated graph:



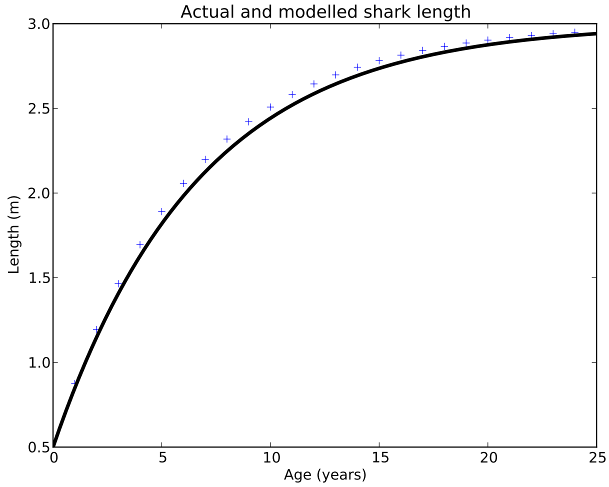
6. Here is a possible solution.

```
# Program to model the length of a shark.
from pylab import *

# Plot the exact solution to the DE
t = arange(0, 25, 0.1)
exactLength = 3-2.5*exp(-0.15*t)
plot(t, exactLength, 'k-', linewidth=3)
show()
```

7. Here are the additional lines of code, inserted before `show()`..

```
# Apply Euler's method for 25 steps with a stepsize of 1
approx=zeros(25)
newTimes=arange(0,25,1)
stepSize = 1
i=0
approx[0] = 0.5
while i<24:
    Ldash = 0.15*(3-approx[i])
    approx[i+1] = approx[i] + stepSize * Ldash
    i = i+1

# Plot the graph
plot(newTimes, approx, 'b+')
xlabel("Age (years)")
ylabel("Length (m)")
title("Actual and modelled shark length")
show()
```

Here is the output from the above program.



(6) 1. Populations cannot grow indefinitely. The growth rate could drop purely because there are not enough resources to maintain the current population growth. Furthermore, as time goes on, countries tend to become more developed. Families in more developed countries typically have higher levels of education, which both reduces the number of children "needed" to help support the family, and also often results in lower birth rates due to deliberate choices and access to contraception.

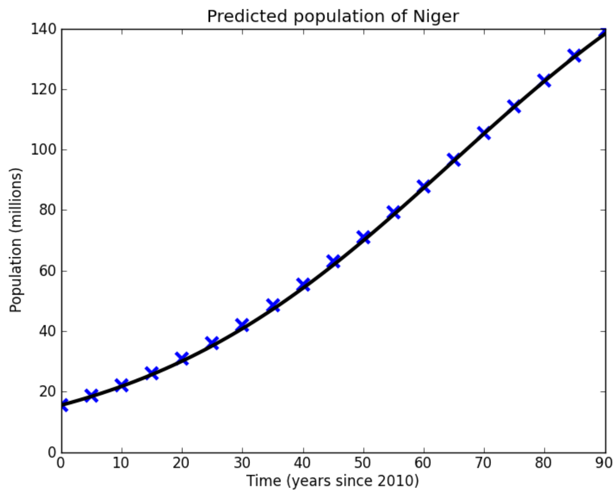
2. This model is reasonable. When P is small compared to C, as the population increases, the term $(1 - P/C)$ is almost 1, so the population increases proportionally with P (what we expect for population growth without constraint). When P is almost as large as C, as the population increases, the term $(1 - P/C)$ gets closer to 0, which means the rate of change of population gets closer to 0. This is the behaviour predicted by the UN, so this is likely to be a reasonable approach.

3. The two missing lines in the program should be:

```
dP = r*P[i] * (1-P[i]/C)
P[i+1] = P[i] + 1*dP
```

An example of a good set of values is:

```
C=190
r=0.038
initPop = 15.5
```



Here is what the output should look like:

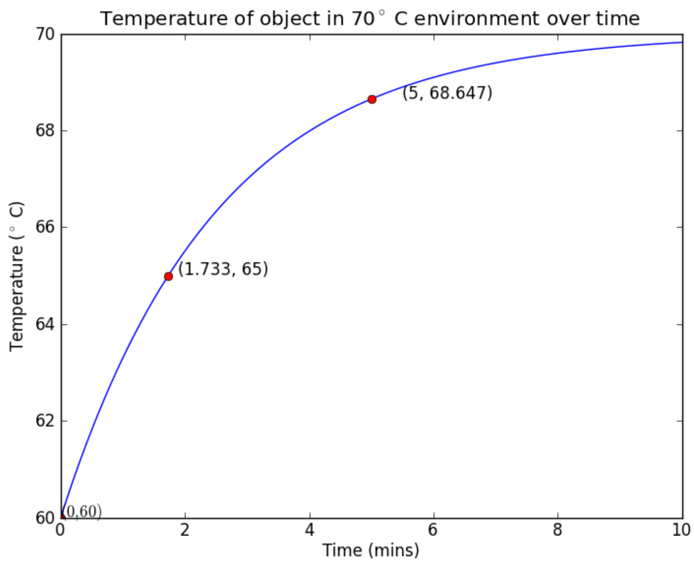
(7) 1. As k is positive, then T' will have the same sign as the term multiplied by k. Furthermore, we expect the temperature of the object to increase (i.e T' to be positive) if the object is colder than the surroundings ($T < C$), and vice versa. $T < C$ implies $0 < C - T$, so T' is positive if $C - T$ is positive, but $T - C$ is negative. So the term with the same sign as T' is $C - T$. Thus, equation 1 is the correct equation.

2. By the hint
$$T'(t) = -10 \times -0.4e^{-0.4t} = 4e^{-0.4t}.$$
Using equation (1), we find that
$$T'(t) = 0.4(70 - (70 - 10e^{-0.4t})) = 0.4(70 - 70 + 10e^{-0.4t}) = 4e^{-0.4t}.$$
These expressions for $T'(t)$ agree, so $T(t) = 70 - 10e^{-0.4t}$ must be a solution to this differential equation.

3. The temperature at $t = 0$ is
$$T(0) = 70 - 10e^{-0.4\times 0} = 70 - 10 = 60\ ^\circ\mathrm{C}.$$
The temperature at $t = 5$ is
$$T(5) = 70 - 10e^{-2} = 68.647\ ^\circ\mathrm{C}.$$
The average rate of change is
$$\frac{\Delta T}{\Delta t} = \frac{68.647 - 60}{5 - 0} = \frac{8.647}{5} = 1.729^\circ\mathrm{C\ min}^{-1}.$$

4. We want to find the time, t, when $65 = T(t) = 70 - 10e^{-0.4t}$. Rearranging gives
$$\begin{aligned} 70 - 65 =& 10e^{-0.4t} \\ \frac{5}{10} =& e^{-0.4t} \\ \log\left(\frac{1}{2}\right) =& -0.4t \\ \frac{\log\left(\frac{1}{2}\right)}{-0.4} =& t \\ 1.733 =& t \end{aligned}$$
so the time when the temperature is $65\ ^\circ$ C is after 1.733 minutes.

5. It will never reach that temperature. For $T(t) = 70$, then $10e^{-0.4t}$ has to be 0, but an exponential will never be 0.

6. 

7. The output is
```
Time  1 64.0 4.0
Time  2 66.4 2.4
Temp is  [ 60.   64.   66.4]
```

8. To change the step size, the lines

```
T[i+1] = T[i] + 1 * Tdash(T[i])
```

and

```
print("Time ",i, T[i], T[i]-T[i-1])
```

should be replaced with

```
T[i+1] = T[i] + 2 * Tdash(T[i])
```

and

```
print("Time ",i*2, T[i], T[i]-T[i-1])
```

To change the number of steps calculated, the lines

```
T = zeros(3)
```

and

```
while i<2:
```

should be replaced with

```
T = zeros(6)
```

and

```
while i<5:
```

(8) Here is the output.

```
Enter value? 7
a:  [ 0.  1.  1.  2.  3.  5.  8.]
b:  [ 0.  1.  1.  0.  1.  1.  0.]
```

(9) The output from Program 1 is:

```
55
10
```

The output from Program 2 is:

```
t = 1
t = 3
t = 8
[  0.   1.   0.   9.   0.  25.   0.   0.  64.   0.]
```

(10) The output from Program 1 is:

```
6
12
24
hello 24 sailors
54
```

Program 2 is an infinite loop, so it never stops running unless interrupted. The output from the program is:

```
0 2
```

(11) The output from the program is:

```
2.9 5
Year  2015 RoC =  0.58
0.5 1
Year  2011 RoC =  0.5
```