SCIE1000 Tutorial sheet 11

This tutorial contributes toward your final grade; see the Course Profile (https://www.uq.edu.au/study/course.html?course_code=SCIE1000). The tutorial will be marked out of 6, with 3 marks for completing the "Before class" work, and 3 marks for completing the "In class" assessment and working on the remaining "In class" questions until you finish them or the tutorial ends.

Goals: This week you will work through some calculation and discussion questions, relating to applications of differential equations to modelling populations and temperature change. As usual, broad concepts and techniques are more important than specific examples. Do not try to commit lots of facts to memory; instead, know **how** to do things, and **when** certain models and approaches are appropriate. Some of the questions on this sheet are taken directly from previous exam papers. There are more questions on the tutorial sheet than you can finish in class; do some of the remaining ones outside class. You should have started preparing for the final exam. Remember that you may take in a double sided sheet of handwritten or typed notes. What materials will you take in? Do you need to re-write any key points in short, easily accessible form? There are no memory questions on the exam, so don't try to commit things to memory. Now is a good time to consolidate your understanding of SCIE1000.

To be completed before class

Complete the following questions before class, write (or type if you wish) your answers on a sheet of paper, put your name and student number on the top of the paper, and hand it to your tutor as you enter the room. **If you do not hand in the answers at the start of the class, as you enter the room, then you will lose the marks for this component.** Note that in some cases there are no "right" or "wrong" answers.

Question (1)

The Population Division of the Department of Economic and Social Affairs of the United Nations periodically releases estimates of the current populations of all countries, and also projections of the population sizes in coming years. In 2015, they released the *2015 Revision* of *World Population Prospects* (https://esa.un.org/unpd/wpp/), which includes predictions of populations until the Year 2100.

1. Read the pages numbered 1-11 of *World Population Prospects: The 2015 Revision - Key Findings and Advance Tables* (https://esa.un.org/unpd/wpp/publications/files/key_findings_wpp_2015.pdf) and take some useful notes about what it says. (There may be questions relating to this on the final exam.) You do not need to note down details, but instead what are some of the key messages? Did you find anything surprising or interesting? What does this mean for mankind, and resources on Earth? Come along to class ready to discuss the answers to these questions.

2. Table 1 in the document includes seven sets of data on current and projected future populations of the world and of six regions. For each of the seven data sets, state what type of equation could be used to model the corresponding data (for example, linear with a positive slope, linear with a negative slope, quadratic curving up, etc). Briefly explain your answers; there is no need to produce specific equations. (Hint: it might help to use Python to graph the data for each region. Note that the years of the predictions are not equally spaced, so plotting graphs is a good idea.)

In [ ]:

```
# Write a Python program here if you wish to see the shapes of the graphs.
```

Question (2)

View the *world population clock* (http://www.worldometers.info/world-population). A lot of the information on that page and linked from that page is interesting and important, but we shall focus on a few aspects.

1. Using the "real time" population clocks, predict the global number of births and deaths that will occur in the current calendar year, and hence predict the increase in global population during the current calendar year. Explain how you obtained your answers.

2. Currently, the population of China is larger than that of India, but India is growing more quickly. Estimate by how much the population of India is "catching" that of China each day. (Hint: you could watch the comparative movements over a short time interval and extrapolate.)

3. While watching the real time population clocks, what did you *think*, and how did you *feel*? Write down a few brief notes.

4. (This question was on the final examination in 2013, and worth 4 marks. Expected working time was about 4 minutes.) By hand, find all of the output produced by the following partial Python program.

```
Pop = array([15.6, 16.1, 18.5, 19.9, 22.1])
n = 0
while n<4:
    if Pop[n] > 18:
        print(n, Pop[n] - Pop[n-1])
    n = n+1
```

To be completed in class

Complete the following questions in class. They involve a mix of individual work, and discussions with others. Make sure that you read the questions before class and think about how you might approach answering them. Don't rely on someone else doing all of the work. You need to work by yourself on the final exam, so it is important that you work hard now.

Feedback: Be proactive!

Australian government research shows that students often feel they don't receive adequate feedback on their work. In a class of 800 students, it is not possible for the course coordinator to give direct feedback to each student. Instead, tutorial classes are designed to be the place in which you can get feedback on your work from classmates and the tutors. You can ask for help, show them your answers, and discuss your understanding of any of the course material. As an adult learner, the onus is on **you** to seek feedback; tutors and classmates are happy to give it, if you want it.

Question (3)

Your tutors will record the marks for the sheet of paper you submitted with the "Before class" work, and they will then return your sheet to you early in the class so you can work from it.

1. Discuss your notes from Question (1) Part 1 with a partner. If they have written something that you think may be important and that you missed, then update your notes.

2. Discuss the answers to Parts 2 and 3 of Question (1) as a group, and update your answers as appropriate.

3. Briefly discuss your answers to Question (2) with your partner, and update your answers as appropriate.

Question (4)

(This question was on the final examination in 2008, and worth 30 marks. Expected working time for this question was about 30 minutes.)

Bob the biologist is modelling the growth of a certain species of algae over a given time period. Let $P(t)$ be the population of algae at any time t in hours, in individuals per mL of water.

1. Bob believes that the population satisfies the differential equation $P' = kP$, where k is a constant. Explain briefly what this equation means. What is the physical meaning of the constant k?

2. Show that $P(t) = Ae^{kt}$ is a solution to the equation in Part (a), where A is a constant. What is the physical meaning of the constant A? (Hint: the derivative of e^{kt} is ke^{kt}.)

3. Recall that $P(t) = Ae^{kt}$. Bob's experiments show that at time $t = 2$ hours, $P(2) = 200$ individuals per mL of water, and at time $t = 6$ hours, $P(6) = 400$ individuals per mL. Find an equation for the population $P(t)$. (Round the value of the constant A to zero decimal places, and the value of k to three decimal places.)

4. Bob asks you whether his model for $P(t)$ in Part 3 is likely to be realistic over an extended time period. Respond to Bob's question, with reasons justifying your answer. (You should include a rough sketch of the algae population over time as predicted by Bob's model. If you believe his model is inaccurate, include a rough sketch of what you believe is a more accurate prediction of the population over time.)

5. An alternate DE for modelling the population of algae is the logistic DE,

   $$P' = kP(1 - \frac{P}{T}).$$

   Explain briefly what this equation means. What is the physical meaning of the term $\left(1 - \frac{P}{T}\right)$?

6. What does the logistic differential equation predict will happen to the population of algae over time if the initial population is $2T$ individuals per mL? Justify your answer briefly with reference to the equation from Part 5.

7. For a particular algae species, $T = 500$ individuals per mL and $k = 0.1$ per hour. At time $t = 0$ hours there are 100 individuals per mL. Apply **one** iteration of Euler's method with a step size of 0.5 hours to predict the population at $t = 0.5$ hours. Show all work.

Question (5)

(This question was on the final examination in 2010, and worth 23 marks. Expected working time for this question was about 23 minutes.)

The *von Bertalanffy growth model* (https://en.wikipedia.org/wiki/Ludwig_von_Bertalanffy#Individual_growth_model) states that the rate of increase in the length $L(t)$ of a shark of age t in years is proportional to an intrinsic positive growth rate r and the difference between a fixed maximum length M and its current length $L(t)$.

1. Write a differential equation (DE) for the length of the shark at any time. (Hint: your answer should be of the form $L'(t) = \ldots$.)

2. Explain carefully what this DE predicts would happen if a mutant shark were born with a length larger than M.

3. Show that

$$L(t) = M - (M - L_0)e^{-rt}$$

   is a solution to the DE in Part 1, where L_0 is the length of the shark at time $t = 0$ when it is born. (Hint: if $y(t) = e^{-rt}$ then $y'(t) = -re^{-rt}$.)

4. A certain shark is born with length equal to $M/2$. At what time does the model predict that the shark will have length equal to M? Justify your answer.

5. For a particular shark, $M = 3$ m, $L_0 = 0.5$ m, t is measured in years and $r = 0.15$ per year.

   A. Find the time at which the shark reaches 2 m in length.
   B. Draw a rough sketch of the length of the shark, for values of t between 0 and 30.

6. The DE in Part 5 has exact solution $L(t) = 3 - 2.5e^{-0.15t}$. In the following Python cell, write a Python program that plots a graph of the exact solution from the time the shark is born until $t = 25$ years.

7. Modify your program from Part 6 so that it also uses Euler's method with a stepsize of one year to approximate the solution to the DE for L' for 25 years. Plot the approximate solution on the same set of axes as the exact solution in Part 6.

In [ ]:
```
# Write your program here
```

Question (6)

This question is a required, in-class assessment piece. To receive the marks for this component, you and your partner must show your answers to a tutor during your tutorial.

The African country Niger has a population of around 15.5 million, and currently has the highest population growth rate of any country in the world. However, the UN predicts that under a medium fertility scenario, the population growth rate will gradually decrease from more than 3.5 % per annum (now) to around 1.2 % per annum in the year 2100.

1. Suggest some physical reasons why the growth rate of Niger might decrease over the next 90 years.

2. Demographers propose using the logistic DE to model the projected population of Niger $P(t)$ over the next 90 years, so

$$P' = rP(1 - P/C)$$

   where r is the growth rate and C is the maximum "carrying capacity" for the population. Is this approach likely to be reasonable? Explain your answer briefly.

3. The following Python program plots the UN projected population of Niger under a medium fertility scenario until the year 2100, at five year intervals. The program also **almost** applies Euler's method to find an approximate solution to the logistic DE proposed in Part 2. However, the lines which perform Euler's method have been removed, and the values of r, C and the initial population in the program are not very useful. Paste the program into the following Python cell, complete the two lines which perform Euler's method, and then experiment with the program to find "good" values for the variables r, C and the initial population (Note that the UN predictions are plotted as blue copies of the letter `x`, and the approximate solution to the logistic DE (once you complete the lines which perform Euler's method) is plotted as a solid black curve.) Show the graphical output of your program to your tutor, along with the final values you chose for the three variables.

```
# Uses Euler's method and the logistic DE to model the population
# of Niger for the next 90 yearxs from 2010.
from pylab import *

# You need to find the next three values
C = 20
r = 0.02
initPop = 40

P = zeros(91)
P[0] = initPop
# Step through Euler's method for 90 years, with stepsize 1 year.
i = 0
while i < 90:
    # Update derivative
    dP = ...

    #Update population
    P[i+1] = P[i] + ...
    i = i+1

### You don't need to change things from here
### This part of the program justs plots the data points and your model.

# Plot the UN predicted pop'n of Niger at 5 year intervals until 2100.
UNx=arange(0,19)*5
UNy=array([15.512,18.500,22.071,26.171,30.841,36.104,41.968,48.423,55.435,
     62.947,70.894,79.203,87.786,96.545,105.372,114.154,122.781,
     131.162,139.209])
plot(UNx,UNy,'bx',mew=3,markersize=10)

# Output approximate solution.
xlabel("Time (years since 2010)")
ylabel("Population (millions)")
title("Predicted population of Niger")
plot(arange(0,91), P, "k-", linewidth=3)
show()
```

In [ ]:

```
# Paste and write your program here
```

Question (7)

(This question was on the final examination in 2012, and worth 28 marks. Expected working time was about 28 minutes.)

Let $T(t)$ be the temperature of an object at time t. When the object is moved to a location in which the temperature is a constant C, the rate of change of the object's temperature at any time is proportional to the difference between T and C.

1. Both $(C-T)$ and $(T-C)$ are expressions for the difference between the temperatures. If k is a **positive constant**, explain briefly but clearly why Equation (1) is a correct representation of the temperature change, but Equation (2) is incorrect.

$$\text{Equation (1)} \quad T' = k(C-T) \qquad\qquad \text{Equation (2)} \quad T' = k(T-C)$$

2. An object with initial temperature 60 $^\circ$C and for which $k = 0.4$ min^{-1} is placed in an oven with a constant temperature 70 $^\circ$C. Show that the following function is a solution to the DE from Equation (1) in Part 1. (Hint: if $y(t) = ae^{bt}$, where a and b are constants, then $y' = abe^{bt}$.)

$$T(t) = 70 - 10e^{-0.4t}.$$

3. Recall that $T(t) = 70 - 10e^{-0.4t}$. Find the **average** rate of temperature change of the object between $t = 0$ and $t = 5$ mins. Include units in your answer.

4. Find the time at which $T = 65$ $^\circ$C.

5. Find the time at which $T = 70$ $^\circ$C and explain your answer.

6. Draw a rough sketch of the graph of $T(t)$ and mark any values that you know from earlier parts of this question.

7. The following partial program uses Euler's method to solve the DE $T' = 0.4(70-T)$, with an initial temperature of 60 $^\circ$C and a step size of 1 minute. Write down all of the output produced by the program.

```
def Tdash(T):
    return (0.4 * (70-T))

T =  zeros(3)
i = 0
T[0] = 60
while i<2:
    T[i+1] = T[i] + 1 * Tdash(T[i])
    i = i + 1
    print("Time ",i, T[i], T[i]-T[i-1])
print("Temp is ",T)
```

8. Show how to modify the previous program so that it now uses five steps of Euler's method and a step size of 2 minutes to estimate the temperature after 10 minutes. (Show which lines of the program need

Extra questions

Here are some extra practise questions, for you to do in class (if you have time), or outside class. You do not need to do them all, but may like to choose some to help with your preparation for the final exam.

Question (8)

(This question was on the final examination in 2010, and worth 7 marks. Expected working time for this question was about 7 minutes.)

By hand, work out all of the output generated by the following Python program **when the user enters the value 7 from the keyboard**.

```
from pylab import *
# The command x % y gives the remainder when x is divided by y.

n = eval(input("Enter value? "))
a = zeros(n)
a[1] = 1
i = 2
while i<n:
    a[i] = a[i-1] + a[i-2]
    i = i + 1
print("a: ",a)
b = a % 2
print("b: ",b)
```

Question (9)

By hand, work out all of the output generated by the following two partial Python programs.

Program 1:

```
def f(x):
    ans = x*5
    return ans

def g(x):
    ans = x+4
    return ans

x = 7
print(f(g(x)))
x = g(x-1)
print(x)
```

Program 2:

```
a=zeros(10)
t = 1
while t < 10:
    a[t] = t*t
    if (t > 3) and (t < 6):
        t = t + 1
    else:
        print("t =", t)
    t = 2+t
print(a)
```

Question (10)

(This question was on the final examination in 2008, and worth 7 marks. Expected working time for this question was about 7 minutes.)

By hand, work out all of the output generated by the following two partial Python programs.

Program 1:

```
i = 6
while i < 45:
    print(i)
    if i>20 and i<30:
        print("hello",i,"sailors")
        i = i + 3
    i = i * 2
print(i)
```

Program 2:

```
i = 0
j = 2
result = zeros(3)
print(i,j)
while i <= 2:
    result[i] = j
    j = j+1
print("end:",result)
```

Question (11)

(This question was on the final examination in 2013, and worth 5 marks. Expected working time for this question was about 5 minutes.)

By hand, work out all of the output generated by the following partial Python program.

```
def RoC(Y, P, n):
    delP = P[n] - P[0]
    delY = Y[n] - Y[0]
    print(delP, delY)
    return delP/delY

Yr = array([2010, 2011, 2015, 2017, 2020])
Pop = array([15.6, 16.1, 18.5, 19.9, 22.1])

n = 2
while n>0:
    r = RoC(Yr, Pop, n)
    print("Year ",Yr[n], "RoC = ",r)
    n = n-1
```