SCIE1000, Solutions to Tutorial Week 12.

(1) Here is an example of a hand-written answer which would receive full marks on an exam. (Pay careful attention to the solution; we didn't hand-write it because we were too lazy to type the answer. Instead, it should give you a good idea of how little you need to write, how 'messy' you can be, and how you can use abbreviations.)

2014 exam Question 1.

1. one dose $= 10^6 + 7\times10^5 = 17\times10^5$ units of virus

10 doses/vial $= 17\times10^6$ virus units.

then 1 litre is $33.8 \times 17\times10^6$ units $= 574.6\times10^6$ units/L

so $\dfrac{574.6\times10^6}{6.02\times10^{23}} \dfrac{\text{units/L}}{\text{units/mol}} = 95.4\times10^{-17}$ mol/L.

2. Many possible answers.

* 1 litre contains 338 ~~doses~~ doses, so approx 300.
* 3 doses/person, so 1 litre ~~doses~~ treats 100 people.
* So if N people have been vaccinated then $\frac{N}{100}$ L of OPV have been used.
* how many people vaccinated? Current global population about 7×10^9. If all vaccinated, 7×10^7 L. If 1/3 vaccinated, about 2×10^7 L.

3. A. If 5 secondary infections arise, then a vaccination rate $>80\%$ on average means $<$ one of the 5 people will be susceptible and get sick, so the disease will die out.

B. $80\%/0.95 = 84.2\%$.

C. For: will save more lives and reduce suffering; gives a "safety margin". Against: expensive and difficult to do. Reduces money to spend elsewhere.

(2) The next two pages contain hand-written asnwers. Again, don't only pay attention to the correct answers; also **think** about how much you will write in your answers, and how you will set out your working.

2014 exam. Question 3.

1. * In Year 0 (1988), 35200 cases, and this is constant P_0 in model $P_0 e^{-kt}$.
* Note: 1993→1997, number approx halved. also 1997→2001, so half life about 4 years.

Rule of 72 says decay rate is approx $\frac{72}{4} = 18\%$

Thus is about -0.2 (decay).

so model is $\underline{35200 e^{-0.2t}}$

2. $t = -13$ (note negative). so $P = 473923$.

Likely cause of large difference: as stated, UNICEF commenced eradication campaign in $\underline{1988}$. Thus the model cannot be extrapolated to before the campaign started.

3. $t = 25$, so $P = 237$. Difference probably due to fact that $P(t)$ is just a model. We would expect the actual figure to vary somewhat from the modelled number, simply due to random (or real world) variation.

2014 exam Question 3 continued.

4. We want $P(t)=1$, so $35200e^{-0.2t}=1$

so $e^{-0.2t}=\frac{1}{35200}$ so $-0.2t=\ln\left(\frac{1}{35200}\right)$

so $-0.2t = -10.47$

so $t = 52.35$, so year $1988+52.35$

$=$ year 2040 (approx).

5. A. The function Pred gives the modelled number of polio cases, calculating $P(t)$ for year t.

B. output: 2462.~~3~~6

111.27

51.0

Total: 2624.62

C. For 3 years, finds the sum of the differences between the actual and predicted number of polio cases for each year, and in total.

D. "For each year, find and sum the difference between actual and predicted polio cases."

(3) This is a discussion question.

(4)
1. For both hydrogen and iodine, the rate at which they are consumed in the reaction is dependent tn the interaction between them (that is, it is proportional to the product of the two concentrations). The constant $-k$ in the first two equations is negative because the amounts of both hydrogen and iodine **decrease** as the reaction proceeds, and the constant $-k$ has the same value in both equations because the reaction requires one molecule of hydrogen for each molecule of iodine.

   For hydrogen iodide, the increase in concentration is also dependent on the interaction between hydrogen and iodine, but is double the rate at which hydrogen and iodine individually are consumed. This is to be expected, as each reacting hydrogen molecule and iodine molecule form **two** hydrogen iodide molecules. The constant in this equation is positive, because the quantity of hydrogen iodide increases as the reaction proceeds.

2. To ensure conservation of mass and matter, the number of atoms in the solution (and hence the overall concentration) must remain constant. Thus, all H_2 and I_2 molecules consumed in the reaction must be converted to HI molecules.

3. Consider the DE for the concentration of hydrogen:

$$[H_2]' = -k[H_2][I_2]$$

   $[H_2]'$ has units of mol L^{-1} s^{-1}, whilst both $[H_2]$ and $[I_2]$ have units of mol L^{-1}. Thus, to ensure that this DE is dimensionally balanced, k must have units of L mol^{-1} s^{-1}.

(5)
1. At equilibrium there is no change in the concentration of reactants and products, so the values of $[H_2]'$, $[I_2]'$ and $[HI]'$ are all equal to zero.

2. Consider the DE for the rate of change of HI at equilibrium:

$$\begin{aligned} [HI]' &= 2k_f\,[H_2][I_2] - 2k_r\,[HI]^2 \\ \implies 0 &= 2k_f\,[H_2][I_2] - 2k_r\,[HI]^2 \\ \implies 2k_f\,[H_2][I_2] &= 2k_r\,[HI]^2 \\ \implies k_f\,[H_2][I_2] &= k_r\,[HI]^2 \end{aligned}$$

   Remember that $2k_f\,[H][I]$ represents the rate of formation of hydrogen iodide in the forward reaction. Hence $-2k_r\,[HI]^2$ represents the rate of decomposition of hydrogen iodide in the reverse reaction. Then at equilibrium the rate of formation of HI must equal the rate of decomposition of HI. (Note that it is possible to rearrange any of the DEs of the system at equilibrium to derive this equation.)

(6)
1. Substituting $z = 2$, $x = 1$ and $y = 1$ into the expression for the equilibrium constant gives

$$K = \frac{[HI]^2}{[H_2][I_2]}$$

   Rearranging the equation in Part (e) gives

$$\begin{aligned} k_f[H_2][I_2] &= k_r[HI]^2 \\ \implies \frac{k_f}{k_r} &= \frac{[HI]^2}{[H_2][I_2]} \\ \implies \frac{k_f}{k_r} &= K \\ \implies K &= \frac{6.3 \times 10^{-2}}{1.8 \times 10^{-3}} \\ &= 35 \end{aligned}$$

2. We know that at a particular temperature, the value of the equilibrium constant is unaffected by initial concentrations. Hence

$$K = 35 = \frac{[HI]^2}{[H_2][I_2]}$$

The hint tells us that $[H_2] = [I_2]$, so

$$35 = \frac{[HI]^2}{[H_2]^2}$$

From the hint we know that the number of molecules in the system remains constant, and using the initial concentrations of all reactants/products gives:

$$\begin{aligned}
[H_2] + [I_2] + [HI] &= [H_2]_0 + [I_2]_0 + [HI]_0 \\
&= 1 + 1 + 0 \\
&= 2 \text{ mol L}^{-1}
\end{aligned}$$

Thus $[H_2] + [I_2] + [HI] = 2$ so $[HI] = 2 - [H_2] - [I_2]$.
Remembering that the concentrations of H_2 and I_2 are equal at all times we can rewrite this expression for $[HI]$ as:

$$[HI] = 2 - 2[H_2]$$

Substituting back into the expression for the equilibrium constant gives:

$$\begin{aligned}
35 &= \frac{(2 - 2[H_2])^2}{[H_2]^2} \\
35[H_2]^2 &= 4 - 8[H_2] + 4[H_2]^2 \\
31[H_2]^2 + 8[H_2] - 4 &= 0
\end{aligned}$$

The hint tells us that $[H_2] = 0.2527$ or -0.5107, which must be 0.2527 mol L^{-1} because concentrations cannot be negative. Hence $[I_2] = 0.2527$ mol L^{-1} and $[HI] = 2 - 2 \times 0.2527 = 1.4946$ mol L^{-1}.

(7) Here is an example of a hand-written answer which would receive full marks on an exam.

(a) (i) constant doubling time

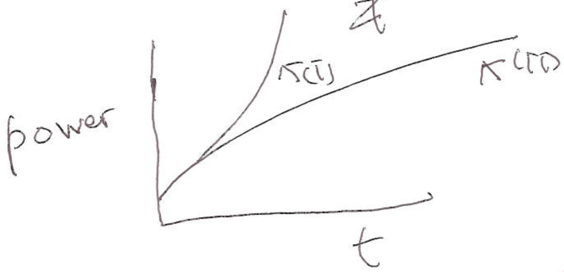
→ exp function

→ eqn (4).

(ii) $\uparrow$ at a $\downarrow$ rate → power function

→ eqn (5), with power $b<1$.

(iii)



(b) (i) logistic growth: eqn (6).

(ii) eqn $y' = ay\left(\frac{b-y}{b}\right)$

rate slower $\Rightarrow$ a $\downarrow$

60% max $\Rightarrow$ b $\downarrow$

(iii)

(8) 1. The expression $\frac{K-R}{K}$ models competition between rabbits, because it models a carrying capacity for the population. As the rabbit population increases, the this term approaches 0, which means their growth rate, R', decreases.

2. The rate of change of the numbers of rabbits and foxes should be in units of rabbits/year and foxes/year. To be able to add and subtract the terms in the equations for R' and F', each term in the equation must have the same units. Thus, we have

$$\text{units}(R') = \text{units}\left(aR\frac{K-R}{K}\right) = \text{units}(bRF)$$

and

$$\text{units}(F') = \text{units}\,(cF) = \text{units}(dRF).$$

Looking at each of these equations individually gives,

$$\begin{aligned}
\frac{\text{rabbits}}{\text{year}} =& \text{units}\,(a)\,\text{rabbits}\frac{\text{rabbits} - \text{rabbits}}{\text{rabbits}} \\
=& \text{units}\,(a)\,\text{rabbits} \\
\text{year}^{-1} =& \text{units}\,(a)
\end{aligned}$$

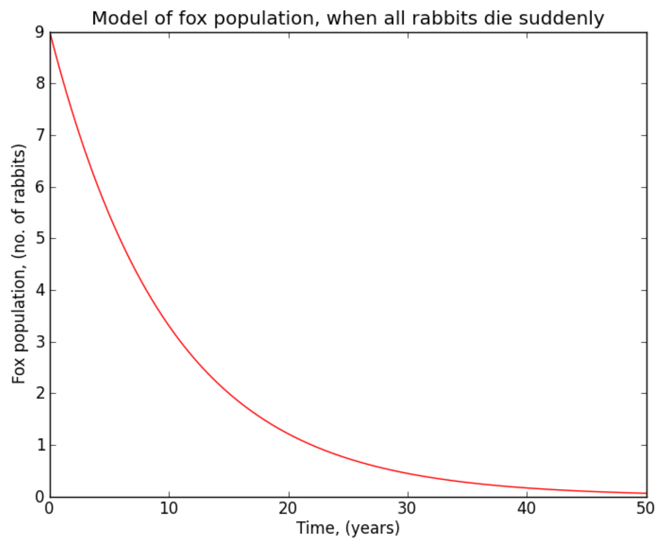
$$\begin{aligned}
\frac{\text{rabbits}}{\text{year}} =& \text{units}(b)\text{rabbits foxes} \\
\text{year}^{-1} =& \text{units}(b)\text{foxes} \\
(\text{year foxes})^{-1} =& \text{units}(b)
\end{aligned}$$

$$\begin{aligned}
\frac{\text{foxes}}{\text{year}} =& \text{units}(c)\text{foxes} \\
\text{year}^{-1} =& \text{units}(c)
\end{aligned}$$

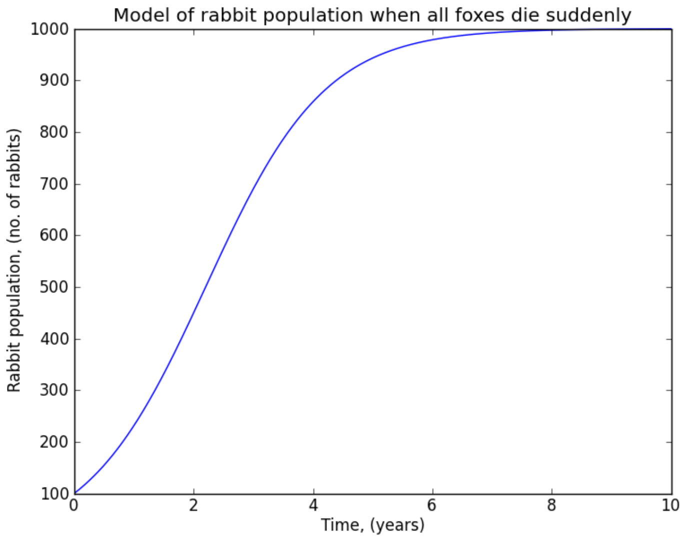
$$\begin{aligned}
\frac{\text{foxes}}{\text{year}} =& \text{units}(d)\text{rabbits foxes} \\
\text{year}^{-1} =& \text{units}(d)\text{rabbits} \\
(\text{year rabbits})^{-1} =& \text{units}(d).
\end{aligned}$$

So the units of a and c are year^{-1}, b are (year foxes)$^{-1}$, and d are (year rabbits)$^{-1}$.

3. If all the rabbits died suddenly, R would become 0. This means that the rate of change of foxes would be modelled by $F' = -cF$. The rate of change of foxes is negative and proportional to the number of foxes, which means that the population of foxes is exponentially decaying. This makes sense, as the foxes would starve to extinction if their main food source becomes extinct. Here is a sketch of this behaviour:



4. If all the foxes died suddenly, F would become 0. This means that the rate of change of rabbits would be modelled by $R' = -aR\left(\frac{K-R}{K}\right)$. This is the logistic model, so we know the population of rabbits will increase towards the carrying capacity K. This makes sense, as without their main predators, the rabbits will be free to live and breed to the point where the environment can just sustain the population. Here is a sketch of this behaviour:



5. Using these values,

$$R' = aR\left(\frac{K-R}{K}\right) - bRF = 1 \times 100 \times \left(\frac{1000-100}{1000}\right) - 0.1 \times 9 \times 100 = 90 - 90 = 0$$

   and

$$F' = -cF + dRF = -.1 \times 9 + 0.001 \times 9 \times 100 = 0.9 - 0.9 = 0.$$

   Even without performing Euler's method, we can see that the populations won't change, since the rates of change of both the fox and rabbit populations are 0. Performing one step of Euler's method anyway gives

$$R = 100 + 2 \times 0 = 100 \quad \text{and} \quad F = 9 + 2 \times 0 = 9.$$

6. We found the rates of change to be 0 in part (e), which means the populations of rabbits and foxes under these conditions will remain constant. This means that for the values described in that question, the model is in perfect equilibrium. There are exactly enough rabbits to feed the foxes, but not allow their population to grow, and there are exactly enough foxes to keep the population of rabbits from increasing, but not eat so many that their population drops.

7. A. Using these values, we can find an expression for the rate of change of the fox population in terms of the rabbit population,

$$F' = -0.1 \times 20 + 0.001 \times 20 \times R = 0.02R - 2.$$

   So if $R = \frac{2}{0.02} = 100$, then $F' = 0$ (i.e. the fox population remains constant). Furthermore, if $R > 100$ then $F' > 0$ (i.e. the fox population increases), and if $R < 100$ then the $F' < 0$ (i.e. the fox population decreases).

   B. Again, using these values, we can find an expression for the rate of change of the rabbit population as a function of the current rabbit population,

$$R' = 1 \times R \times \left(\frac{1000-R}{1000}\right) - 0.1 \times 20 \times R = R\,(1 - 0.001R - 2) = -R(1 + 0.001R).$$

   Since the rabbit population is always positive, the term $R(1 + 0.001R)$ is always positive, so R' will be negative, regardless of the actual value of R.

(9) Here is an example of a hand-written answer which would receive full marks on an exam.

(a)

$$\begin{tikzcd} E \arrow[r, "0.05"'] & L \arrow[r, "0.1"'] & P \arrow[r, "0.6"'] & A \arrow[lll, bend right, "1000"'] \end{tikzcd}$$

(b)

$$E' = 1000A - E$$

$$L' = 0.05E - L$$

$$P' = 0.1L - P$$

$$A' = 0.6P - A.$$

note: 1000 because of 2 adults, one male and one female → avge 1000 eggs each.

(c) at $t=0$, $E' = 1000 \times 2 - 100$
$= 1900.$

$$L' = 0.05 \times 100 - 50 = -45$$

$$P' = 0.1 \times 50 - 10 = -5$$

$$A' = 0.6 \times 10 - 2 = 4.$$

Hence at $t=1$,

$$E = 100 + 1900 \times 1 = 2000.$$

$$L = 50 - 45 \times 1 = 5$$

$$P = 10 - 5 \times 1 = 5$$

$$A = 2 + 4 \times 1 = 6$$