SCIE1000 Tutorial sheet 13

This tutorial contributes toward your final grade; see the Course Profile (https://www.uq.edu.au/study/course.html?course_code=SCIE1000). The tutorial will be marked out of 6, with 3 marks for completing the "Before class" work, and 3 marks for completing the "In class" assessment and working on the remaining "In class" questions until you finish them or the tutorial ends.

Goals: This tutorial sheet contains questions covering systems of differential equations, from previous SCIE1000 exams. Take the exam seriously and make sure you approach your study in the right way. You will need to be quite familiar with the approaches we have covered throughout the course, and be able to think about problems and situations that you haven't encountered before. Also, the exam is quite long. You will need to work quickly, make rapid decisions about models, justify your choices, interpret answers and draw conclusions. Previous exam questions give a good idea of what to expect. Also check the online *marks recording system* (https://www.maths.uq.edu.au/marks), to ensure that all of your marks have been recorded.

As we did last week, on this tutorial sheet we also will work through the exam paper from 2014.

To be completed before class

Complete the following questions before class, write (or type if you wish) your answers on a sheet of paper, put your name and student number on the top of the paper, and hand it to your tutor as you enter the room. **If you do not hand in the answers at the start of the class, as you enter the room, then you will lose the marks for this component.** Note that in some cases there are no "right" or "wrong" answers.

Polio

Read the following information, which appeared as the final page of the 2014 examination paper.

> **Background:** *Polio* is a highly contagious infectious viral disease, transmitted from an infected host to a susceptible individual via the faecal-oral route. Most people who suffer from polio will recover completely, but a small proportion will develop long lasting paralysis. Polio existed for thousands of years, but major epidemics did not occur until late in the 19th century. In the 20th century, major polio epidemics caused many thousands of children and adults to become paralysed, including in first world countries, and promoted widespread fear.
>
> **Epidemiology:** The typical infectious period for an individual with polio is 2 weeks, and on average 5 secondary infections will arise from a single infected individual in a fully susceptible population. In about 99 % of cases, an individual infected with polio will recover completely with no long term effects. However, in about 1 % of cases an infected individual will develop long term paralysis from the disease.
>
> **Polio eradication:** The possibility of developing paralysis makes polio a highly feared disease. In 1988, the World Health Organization, UNICEF and the Rotary Foundation commenced a global campaign to eradicate polio. The following table shows the global number of reported polio cases per year in a selection of years since 1988.

Year	Years since 1988	Number of recorded cases per year
1988	0	35251
1990	2	23484
1993	5	10487
1997	9	5185
2002	14	1922
2007	19	1387
2012	24	291

Vaccination: A cornerstone of the global polio eradication campaign is widespread vaccination, using the Oral Polio Vaccine (OPV) developed by Albert Sabin in 1957. There are three *serotypes* or variations of the polio virus, called *Sabin 1*, *Sabin 2* and *Sabin 3*.
A single OPV dose (usually two drops) contains 10^6 infectious units of the Sabin 1 strain, 10^5 infectious units of the Sabin 2 strain, and 6×10^5 infectious units of the Sabin 3 strain. Three doses of OPV produce protective antibodies to all three polio serotypes in about 95 % of recipients.

Polio test: In 1994-1996, the WHO conducted a trial to check the effectiveness of its world wide network of laboratories in which they test samples for polio. They prepared 500 uninfected biological samples, and then infected 61.2 % of these samples with at least one strain of polio virus. To make the test harder, many samples were infected with viruses similar to polio, but which were not actually polio. They sent these samples to 67 of their labs across the world, and each lab attempted to identify which of the samples they had been sent were infected with polio. Overall, the tests for polio undertaken in the network of laboratories had a sensitivity of 92 % and a specificity of 90 %.

Question (1)

By this stage in the semester, you should have started studying for the SCIE1000 exam. The following is a question that appeared on the 2014 SCIE1000 final exam. Attempt to do this question under "pretend" exam conditions, and see how long it takes you. Marks that were allocated on the exam are shown, for your reference. Working time was about 1 minute per mark. You do not need to get the answer correct in order to receive the tutorial marks. Instead, your work needs to demonstrate that you have made a serious attempt at answering the question.

Using the information about polio given above, answer the following questions which formed Question 6 on the 2014 exam paper.

Consider a stage-structured model of a polio epidemic, with time step size one week and with four stages: Susceptible people S; Infected people I; Removed people R, who have either fully recovered with no lasting effects **or** who are immune to polio; and Paralysed people P, who have developed long-lasting paralysis from having polio. To reduce the severity of the epidemic, public health officials vaccinate 10 % of all susceptible people each week, however the vaccine is only 95 % effective at inducing immunity.

Run the Python program in the following cell, which draws a life cycle diagram for a polio epidemic.

1. (**7 marks**) Write a system of differential equations for S', I', R' and P'. Show all working.

2. (**2 marks**) In your equations, you should have $S' + I' + R' + P' = 0$. Explain briefly what this means, and why it is expected.

3. (**3 marks**) One individual infected with polio enters a completely susceptible population of 100,000 people. Table 1 (below) shows the results of using Euler's method to predict S, I, R and P at the **start** of various weeks from Week 1 until Week 30. (Note: values are rounded to whole numbers.) **From the table**, **not** the equations, find the values of S', I' and R' for the first week.

4. (**6 marks**) Using the data from Table 1, write a brief, word-based summary of the modelled polio epidemic, to help public health authorities with their planning.

5. (**4 marks**) In the epidemic summarised in Table 1, the first week in which $P' \neq 0$ is Week 6. Explain, in words, why it would be expected that $P' = 0$ for a number of weeks at the start of the polio epidemic.

6. (**6 marks**) Consider **all** weeks of the epidemic, not only the weeks summarised in Table 1. Will the values of each of S', I', R' and P': always be ≥ 0; or always be ≤ 0; or neither (that is, it will take on a range of positive, 0 and negative values). In each case, identify the correct option, and justify your answer.

   - S' is: always ≥ 0 or always ≤ 0 or neither. Why?
   - I' is: always ≥ 0 or always ≤ 0 or neither. Why?
   - R' is: always ≥ 0 or always ≤ 0 or neither. Why?
   - P' is: always ≥ 0 or always ≤ 0 or neither. Why?

Week	S	I	R	P		**Week**	S	I	R	P

Week	S	I	R	P		Week	S	I	R	P
1	100000	1	0	0		10	39134	1062	59798	7
2	90498	3	9500	0		15	17638	3105	79199	59
3	81894	8	18099	0		20	7567	1102	91211	121
7	54667	205	45127	1		30	2551	9	97303	138

In [ ]:

```
# Run this program to draw a life cycle diagram for a polio epidemic.

from pylab import *

plot([1,2,3,3],[1,1,1.4,0.6], 'o', ms=45, markerfacecolor="None", markeredgeco
lor='red', markeredgewidth=1)
text(0.94,0.97,"S")
text(1.94,0.97,"I")
text(2.94,1.37,"R")
text(2.94,0.57,"P")

arrow(1.28, 1.0, 0.33, 0, head_width=0.1, head_length=0.1, fc='k', ec='k')
arrow(2.28, 1.0, 0.35, 0.3, head_width=0.1, head_length=0.1, fc='k', ec='k')
arrow(2.28, 1.0, 0.35, -0.3, head_width=0.1, head_length=0.1, fc='k', ec='k')

arrow(1.28, 1.0, 0.72, 0.45, head_width=0, head_length=0, fc='k', ec='k')
arrow(2, 1.45, 0.6, 0, head_width=0.1, head_length=0.1, fc='k', ec='k')

text(0.9,1.86,"Life cycle diagram for polio epidemic")


plot([0,4],[0,2],linewidth=0)
axis('off')
show()
```

Question (2)

You are permitted to take a double sided sheet of paper into the final exam, comtaining any handwritten or typed material ypu would like. Write down a brief overview of what you might include on that sheet of paper. (Do not write the detailed material itself, instead write things such as "general equation for a sine function".)

To be completed in class

Complete the following questions in class. They involve a mix of individual work, and discussions with others. Make sure that you read the questions before class and think about how you might approach answering them. Don't rely on someone else doing all of the work. You need to work by yourself on the final exam, so it is important that you work hard now.

Feedback: Be proactive!

Australian government research shows that students often feel they don't receive adequate feedback on their work. In a class of 800 students, it is not possible for the course coordinator to give direct feedback to each student. Instead, tutorial classes are designed to be the place in which you can get feedback on your work from classmates and the tutors. You can ask for help, show them your answers, and discuss your understanding of any of the course material. As an adult learner, the onus is on **you** to seek feedback; tutors and classmates are happy to give it, if you want it.

Question (3)

The tutors will record your marks for completing the "Before class" work, and will immediately return your sheet to you. Discuss your answer to Question (2) with a partner. If they have written something that you think may be important and that you missed, then update your notes. This will help you prepare for sitting the final exam!

Question (4)

1. Tutors will work through the correct answers to Question (1) on the board. Discuss the answers with a partner if there is anything you do not understand. When you are confident that you understand the answers, **without discussing it with anyone else**, grade your own work: give yourself a mark for each part of your answer to Question (1), and hence a total mark out of 28.

2. Mark your partner's work **without discussing it with them and without letting them see the marks you assigned yourself in Part 1**.

3. When you have both finished, discuss with your partner the marks you assigned to your work and their work in Parts 1 and 2. Try to come to a resolution on any differences, and ask a tutor if you need clarification.

Question (5)

This question is a required, in-class assessment piece. To receive the marks for this component, you and your partner must show your answers to a tutor during your tutorial.

(This question was on the final examination in 2013, with the number of marks for each part as shown. Expected working time for this question was about one minute per mark.)

The total number of live births in the country Niger in 2010 was 700,000. The total population of Niger fell into the following age categories.

	Age 0-14	**Age 15--49**	**Age 50+**
Total population in age group in 2010 (thousands)	7600	6600	1400
Annual number of deaths in age group in 2010 (thousands)	122	30	44

Consider a stage-structured model of the population of Niger, with the following three stages: Young people aged 0-14, denoted Y; Adult people aged 15-49, denoted A; and Older people aged 50+, denoted O. Run the Python program in the following cell, which draws two life-cycle diagrams with a step size of one year.

1. (**9 marks**) Briefly explain or demonstrate how to calculate each of the "transition" values in the first life-cycle diagram. (Hint: assume that all new parents are in the adult stage, and note that individuals spend 15 years in the Y stage and 35 years in the A stage. Remember that the step size is one year. Values have been rounded.)

2. (**6 marks**) The system of differential equations corresponding to the life-cycle diagram is

$$Y' = -0.083Y + 0.106A$$

$$A' = 0.067Y - 0.034A$$

$$O' = 0.029A - 0.031O$$

   Use one step of Euler's method and a stepsize of **two years** to predict the population of Niger in each stage in 2012.

3. (**3 marks**) Applying 45 steps of Euler's method would give a prediction of the Nigerien population in the year 2100. Would you expect this prediction to be similar to, greater than or less than the UN prediction of the Nigerien population in 2100? Briefly justify your answer.

4. The second life-cycle diagram for Niger matches the first one, except that the transition probabilities and numbers have been replaced by letters $a, b, \ldots, f$. In each of the following hypothetical scenarios indicate which (if any) of $a, b, \ldots, f$ would be likely to be larger in value than the corresponding value(s) in Part 1, and which (if any) would be likely to be smaller. Briefly justify your answers.

   A. (**2 marks**) There is a very successful national education campaign promoting contraception.

   B. (**2 marks**) A highly contagious sexually transmitted disease rapidly mutates to a virulent new form, with a much higher death rate.

C. (**2 marks**) There are massive nation-wide improvements in public health and nutrition.

In [ ]:
```
# Run this program to draw life cycle diagrams for the population of Niger.

from pylab import *

def skel():
    plot([1,2,3],[1,1,1], 'o', ms=45, markerfacecolor="None", markeredgecolor=
'red', markeredgewidth=1)
    text(0.94,0.97,"Y")
    text(1.94,0.97,"A")
    text(2.94,0.97,"O")

    arrow(1.0, 0.8, 0, -0.3, head_width=0.1, head_length=0.1, fc='k', ec='k')
    arrow(2.0, 0.8, 0, -0.3, head_width=0.1, head_length=0.1, fc='k', ec='k')
    arrow(3.0, 0.8, 0, -0.3, head_width=0.1, head_length=0.1, fc='k', ec='k')

    arrow(1.28, 1.0, 0.33, 0, head_width=0.1, head_length=0.1, fc='k', ec='k')
    arrow(2.28, 1.0, 0.33, 0, head_width=0.1, head_length=0.1, fc='k', ec='k')

    arrow(2, 1.2, -0.45, 0.3, head_width=0.0, head_length=0.0, fc='k', ec='k')
    arrow(1.55, 1.5, -0.45, -0.24, head_width=0.1, head_length=0.1, fc='k', ec
='k')

    plot([0,4],[0,2],linewidth=0)
    axis('off')

skel()
text(1.4,1.55,"0.106")
text(1.3,0.8,"0.067")
text(2.3,0.8,"0.029")
text(0.8,0.27,"0.016")
text(1.8,0.27,"0.005")
text(2.8,0.27,"0.031")
text(1.3,1.86,"First life cycle diagram")
show()

skel()
text(1.5,1.55,"d")
text(1.45,0.8,"b")
text(2.45,0.8,"e")
text(0.95,0.27,"a")
text(1.95,0.27,"c")
text(2.95,0.27,"f")
text(1.3,1.86,"Second life cycle diagram")
show()
```

Question (6)

Check the online *marks recording system* (https://www.maths.uq.edu.au/marks), to ensure that all of your marks have been recorded. If you believe that marks are missing, email the course corodinator.

Question (7)

This question uses the information about polio from the 2014 examination paper, given above. Answer the following questions which formed Question 2 on the 2014 exam paper; marks that were allocated on the exam are shown, for your reference. Working time was about 1 minute per mark.

Answer the following questions, with reference to the sensitivity and specificity of the 500 polio tests conducted in 1994-1996 on laboratory-prepared samples, 61.2 % of which had been deliberately infected with polio. Consider the following table:

	Sample infected	**Sample not infected**
Laboratory test positive	A	B
Laboratory test negative	C	D

1. (**6 marks**) Calculate the values of A, B, C and D in the table. Show your working.

2. (**2 marks**) Assume that a particular sample in this study returned a positive test for polio. Find the probability p that this test result was a true positive.

3. (**5 marks**) Assume that in 1994, a sample from a person chosen at random from the global population tested positive for polio in a WHO laboratory. Is this result more likely to be a true positive than the value of p you calculated in Part 2, or less likely, or the same? Justify your answer.

Question (8)

This question uses the information about polio from the 2014 examination paper, given above. Answer the following questions which formed Question 4 on the 2014 exam paper; marks that were allocated on the exam are shown, for your reference. Working time was about 1 minute per mark.

Consider a graph of the number of polio cases reported each year. The Area Under the Curve (AUC) of that graph between two years estimates the **total** number of cases reported during that time period.

1. (**5 marks**) The table given in the polio information above shows the number of reported polio cases per year for a selection of years. Assume that those data are plotted as points on a graph, with straight lines connecting consecutive points. Find the **exact** AUC of the resulting graph between $t = 0$ (the year 1988) and $t = 24$ (the year 2012). There is no need to plot the graph, unless you wish to. (Hint: to simplify your calculations, the AUC from 1997 to 2012 is 30235.)

2. (**4 marks**) According to UN data, the total number of reported polio cases between 1988 and 2012 was around 178300. Calculate the percentage error in your estimate from Part 1 compared to the real figure, and comment briefly.

3. (**6 marks**) On the tutorial sheet last week, the function $P(t) = 35200e^{-0.2t}$ was used to model the annual number of reported polio cases, where t is the number of years since 1988. Assuming that $P(t)$ will remain a good model into the future, calculate an estimate of the total number of polio cases that will *ever* occur from 2013 onwards. Show all working. (Hint: $\int 35200e^{-0.2t}\ dt = -176000e^{-0.2t} + C$.)

4. (**5 marks**) Is your answer to Part 3 likely to be correct in reality? Explain why or why not.

Question (9)

This question uses the information about polio from the 2014 examination paper, given above. Answer the following questions which formed Question 5 on the 2014 exam paper; marks that were allocated on the exam are shown, for your reference. Working time was about 1 minute per mark.

Before mass vaccination, polio occurred in many countries. Run the Python program in the following cell, which plots a graph showing $\log_{10}$ of the mean number of polio cases reported each week of the year in the Netherlands between 1946 and 1957.

1. (**6 marks**) Find a function $L(t)$ using $\sin$ that models the data in the graph, where t is the number of the week in the year. Explain briefly how you obtained $L(t)$.

2. (**4 marks**) Which week numbers have the smallest and largest mean number of reported cases? Find the mean number of cases in each of those weeks. (Hint: note that the graph shows $\log_{10}$.)

In [ ]:
```
from pylab import *

t=arange(0,52)
P=array([2.9,2.9,2.8,3.4,2.8,1.3,1.25,1.2,0.8,1,0.7,0.9,0.6,0.7,0.8,1.1,1.3,2.
1,
         1.4,2,3.5,5,6,9,10,12,19,21,30,33,32,28,34,29,31,29,28,27,26,22,25,
         19.2,14,18,11,8,6,8,3.9,5.2,3.3,3])

plot(t,log10(P),'kx',mew=3,markersize=8)
xlabel("Week of the year")
ylabel("log10 of (mean reported cases)")
title("log10 of reported polio cases")
grid(True)
show()
```

Extra questions

Here are some extra practise questions, for you to do in class (if you have time), or outside class. You do not need to do them all, but may like to choose some to help with your preparation for the final exam.

Question (10)

(This question is based on a question from the final examination in 2010.)

Researchers are trialling a new treatment for a form of cancer. Each month:

- 60 people are newly diagnosed with cancer
- a person with cancer will: move into remission (30 % likelihood); die (20 %); or continue with cancer (and treatment) for another month (50 %).
- a person in remission will: have their cancer redevelop (10 % likelihood); die (5 %); or continue in remission (85 %).

Researchers wish to develop a model based on a system of differential equations.

1. Draw a life-cycle diagram with two stages, C representing people with cancer and R representing people in remission.

2. Let $C(t)$ and $R(t)$ be the number of people with cancer and in remission at any time t in months. Explain **briefly** why the following pair of differential equations models the situation described above.

$$C' = 60 - 0.5C + 0.1R \qquad\qquad R' = 0.3C - 0.15R$$

3. At time $t = 0$, 300 people have cancer and 40 more are in remission. Apply one step of Euler's method with a step size of 0.5 month to predict the values of both C and R after 0.5 month. Show all work.

4. Run the Python program in the following cell, which applies Euler's method for a period of 24 months, and plots a graph.

   A. In order to plan for providing ongoing health support, the government seeks your advice on the likely long-term number of people in each stage. What advice would you provide? Justify your answer.

   B. A biologist proposes modelling $R(t)$ in the long term by a power function, with the power between 0 and 1. Briefly discuss the effectiveness of this approach.

5. By hand, find all of the output produced by the following program, which applies two steps of Euler's method to this pair of equations.

```
def Cdash(Cval, Rval):
    return( 60 - 0.5 * Cval + 0.1 * Rval)

def Rdash(Cval, Rval):
    return( 0.3 * Cval - 0.15 * Rval)

step = 0.5
maxStep = 2
t = zeros(maxStep+1)
C = zeros(maxStep+1)
R = zeros(maxStep+1)
C[0] = 300
R[0] = 40
i = 0
```

In [ ]:

```
# Euler's method to solve DE on the 2010 exam paper.
from pylab import *

def Cdash(Cval, Rval):
    return( 60 - 0.5 * Cval + 0.1 * Rval)

def Rdash(Cval, Rval):
    return( 0.3 * Cval - 0.15 * Rval)

step = 1
maxStep = 24
t = arange(0,maxStep+1)
C = zeros(maxStep+1)
R = zeros(maxStep+1)
C[0] = 300
R[0] = 40
i = 0

while i < maxStep:
    CD = Cdash(C[i], R[i])
    RD = Rdash(C[i], R[i])
    C[i+1] = C[i] + step * CD
    R[i+1] = R[i] + step * RD
    i = i+1
plot(t, C, 'k-', linewidth=3)
plot(t, R, 'k-', linewidth=3)
xlabel('Time (months)')
ylabel('Number of people')
title('Cacncer treatment model')
text(10,150,'C(t)')
text(11,330,'R(t)')
show()
```